Exercise Sheet 5

Exercise 1. In this exercise we will show that the Riemann surface $\mathbb{CP}^1$ is a $SU(2)$-homogeneous space.

(1) Show that the action of $SU(2)$ on $\mathbb{CP}^1$ defined by $g \cdot [v] = [gv]$ is transitive.
(2) Show that

$$\mathrm{Stab}_{SU(2)}([1:0]) = \left\{ \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} : \theta \in \mathbb{R} \right\} \cong S^1,$$

and deduce that

$$\Psi : SU(2)/S^1 \to \mathbb{CP}^1, \quad gS^1 \mapsto g \cdot [1:0]$$

is well-defined and a bijection.
(3) Pull back the complex structure of $\mathbb{CP}^1$ along Ψ to $SU(2)/S^1$ and conclude that Ψ is a biholomorphism.

Exercise 2. Let $C_P \subset \mathbb{P}^2_{\mathbb{C}}$ be a smooth projective curve defined by a homogeneous polynomial $F(X, Y, Z) \in \mathbb{C}[X, Y, Z]$. Consider the standard affine charts $U_Z = \{Z \neq 0\}$ and $U_Y = \{Y \neq 0\}$. Suppose $[x_0 : y_0 : 1] \in C_P \cap U_Z$ is a smooth point with $\partial P_y(x_0, y_0) \neq 0$, where $P(x, y) = F(x, y, 1)$.

(1) Show that the map $x \mapsto [x : y(x) : 1]$ defines a local holomorphic parametrization of C_P near $[x_0 : y_0 : 1]$.
(2) Express the parametrization from part (1) in coordinates of the affine chart U_Y.

Exercise 3. Let $U_X = \{X \neq 0\}$, $U_Y = \{Y \neq 0\}$, $U_X = \{Z \neq 0\}$ and identify $\mathbb{C}^2$ with U_2 via $(x, y) = (X/Z, Y/Z)$. For $P(x, y) \in \mathbb{C}[x, y]$, let $\overline{C_P} \subset \mathbb{P}^2_{\mathbb{C}}$ be the Z-homogenized projective closure.

(1) Let $P(x, y) = x^2 + y^2 - 1$. Find the homogeneous equation of $\overline{C_P}$ and show that $\overline{C_P} \cap U_i$ is biholomorphic to $\mathbb{C}\backslash\{0\}$ for $i = X, Y, Z$.
(2) Let $P(x, y) = y^2 - \prod_{j=1}^{n}(x - x_j)$ with distinct $x_j \in \mathbb{C}$ and $n > 2$. Find the homogeneous equation of $\overline{C_P}$ and determine $\overline{C_P} \backslash C_P$.

Exercise 4. (for credit, due on 19 October) *(5 points)* Let $d \geq 2$. Let C_P denote the Fermat curve of degree d in $\mathbb{P}^2_{\mathbb{C}}$ defined by the homogeneous polynomial $X^d + Y^d + Z^d = 0$. Let

$$f : C_P \to \mathbb{P}^1_{\mathbb{C}}, \quad [X : Y : Z] \mapsto [X : Y].$$

Compute the branch values of f.